NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

REPORT No. 158

MATHEMATICAL EQUATIONS FOR HEAT CONDUCTION IN THE FINS OF AIR-COOLED ENGINES

By D. R. HARPER 3d, and W. B. BROWN

WASHINGTON
GOVERNMENT PRINTING OFFICE
1923

AERONAUTICAL SYMBOLS.

1. FUNDAMENTAL AND DERIVED UNITS.

	Symbol.	Metric.		English.	
		Unit.	Symbol.	Unit.	Symbol.
Length	l	meter	m.	foot (or mile)	ft. (or mi.).
Time	t	second	sec.	second (or hour)	sec. (or hr.).
Force	F	weight of one kilogram	kg.	weight of one pound	lb.
Power	P	kg.m/sec		horsepower	IP
Speed		m/sec	m. p. s.	mi/hr	M. P. H.

2. GENERAL SYMBOLS, ETC.

Weight, $W=mg$.

Standard acceleration of gravity, $g=9.806\text{m/sec.}^2=32.172\text{ft/sec.}^2$

Mass, $m=\frac{W}{g}$

Density (mass per unit volume), ρ

Standard density of dry air, 0.1247 (kg.-m.-sec.) at 15.6°C. and 760 mm. = 0.00237 (lb.-ft.-sec.)

Specific weight of "standard" air, 1.223 kg/m.^{3} = 0.07635 lb/ft.^{3}

Moment of inertia, mk^2 (indicate axis of the radius of gyration, k, by proper subscript).

Area, S; wing area, S_w, etc.

Gap, G

Span, b; chord length, c.

Aspect ratio $=b/c$

Distance from $c.\ g.$ to elevator hinge, f.

Coefficient of viscosity, μ.

3. AERODYNAMICAL SYMBOLS.

True airspeed, V

Dynamic (or impact) pressure, $q=\frac{1}{2}\rho V^2$

Lift, L; absolute coefficient $C_L=\frac{L}{qS}$

Drag, D; absolute coefficient $C_D=\frac{D}{qS}$.

Cross-wind force, C; absolute coefficient

$$C_c=\frac{C}{qS}.$$

Resultant force, R

(Note that these coefficients are twice as large as the old coefficients L_c, D_c.)

Angle of setting of wings (relative to thrust line), i_w

Angle of stabilizer setting with reference to thrust line i_t

Dihedral angle, γ

Reynolds Number $=\rho\frac{Vl}{\mu}$, where l is a linear dimension.

e. g., for a model airfoil 3 in. chord, 100 mi/hr., normal pressure, 0°C: 255,000 and at 15.6°C, 230,000;

or for a model of 10 cm. chord, 40 m/sec., corresponding numbers are 299,000 and 270,000.

Center of pressure coefficient (ratio of distance of C. P. from leading edge to chord length), C_p.

Angle of stabilizer setting with reference to lower wing. $(i_t-i_w)=\beta$

Angle of attack, α

Angle of downwash, ϵ

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REPORT No. 158

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SUMMARY.

At the request and with the support of the Engineering Division of the Air Service, United States Army, McCook Field, the Bureau of Standards undertook some laboratory investigations dealing with air-cooled aviation engines, the results of which were submitted to the Committee on Power Plants for Aircraft and by that committee recommended for publication as a technical report of the National Advisory Committee for Aeronautics. In connection with laboratory measurements of the heat-dissipating power of typical engine cylinders, a mathematical analysis of fin behavior was made and is given in this communication.

The introduction contains a description of the paper which will assist the general reader who is not interested in mathematical detail in finding those parts of the paper most likely to prove useful to him. A recapitulation of the mathematical developments is given in Section IV and forms the statement of conclusions reached in so far as a mathematical paper of this type may be said to have conclusions. Numerical examples illustrative of these conclusions are then given, followed by a very brief suggestion of possible application of the equations.

The problem considered is that of reducing actual geometrical area of fin-cooling surface, which is, of course, not uniform in temperature, to equivalent "cooling" area at one definite temperature, namely, that prevailing on the cylinder wall at the point of attachment of the fin. This makes it possible to treat all the cooling surface as if it were part of the cylinder wall and 100 per cent effective.

The quantities involved in the equations are the geometrical dimensions of the fin, thermal conductivity of the material composing it, and the coefficient of surface heat dissipation between the fin and the air stream. Several assumptions of physical nature are thus necessarily involved in making the problem possible of solution. These are set forth in detail, and the limitations which result from them in applying the equations to numerical calculations are carefully pointed out.

An expression for approximate fin effectiveness is developed, based upon simple mathematics and very convenient in form for engineering use. The essence of the paper is an examination into the magnitude of the errors involved in using this expression without correction and a determination of the corrections needed for accurate work. The mathematical expressions involved are quite complicated, including Fourier's Series, super Fourier's Series, Bessel functions of zero order of two kinds with imaginary arguments, etc. The results of the work are collected in graphical form in a series of charts, so that the design engineer can use the simple formula first developed and apply to it corrections readily read from the charts, thus avoiding entirely all higher mathematics.

I. INTRODUCTION.

The equations which express the flow of heat in a metal in terms of simple physical properties are perfectly definite and adapted to numerical computations, although usually somewhat cumbersome and tedious. In applying these equations to the fins on the outside of the cylinders of air-cooled internal-combustion engines, the chief obstacle to numerical work is the great uncertainty of the value to be assigned to one important physical quantity, the rate of dissipation of heat from the fin surface into air under the conditions surrounding the fin. This important deficiency seems to have discouraged any widespread use of the equations of heat conduction in considering the problem, since deductions made from them could be trusted only within rather wide limits.

With the increasing knowledge of rate of cooling in an air stream, it has become more worth while to compile the information obtainable from a mathematical analysis of the problem. The details of such an analysis are not of sufficient general interest to warrant the average reader in following them closely, but it is believed desirable to render them available for reference by those who are working in the same field. The equations which have to do with this subject are bulky, the algebra and integrations tedious and time consuming, and the chances for error are high, although no especially intricate or abstruse reasoning is involved, nor is there much difficulty in interpreting final results other than the necessity of a careful examination of the relative numerical size of the various terms.

With a full appreciation of the tax imposed on a reader by reason of the foregoing facts, the authors have prepared this paper in a form which endeavors to meet the following specification:

(*a*) To segregate, for the benefit of all who are interested in the general subject of air-cooled engines, a general skeleton of the analysis, including the discussion of conditions which bear upon the problem, statement of the exact assumptions to which are applied the mathematical development and the conclusions resulting therefrom, with a few examples of numerical computations to illustrate the practical application of the mathematics.

(*b*) To interline with the above, in such form that the general reader may skip it without losing the thread of the development, such details of mathematical transformation as will be needed by the specialist to reproduce the equations or use them to advantage in their application to his particular problem.

(*c*) To omit all details of algebra, integration, arithmetic, etc., which are merely the mechanism of the mathematical development. Although these steps are essential to an acceptance of the validity of any of the deductions, it must be left to the critical reader to supply the gaps, because the paper is sufficiently complicated in meeting specification (*b*) without such additional weighting.

The basic principle of design which characterizes an air-cooled engine is the providing of some means to increase greatly the natural surface by additional cooling surface, the purpose being to keep the engine cylinder wall temperature down to a value below the upper limit set by satisfactory engine performance. This additional surface takes the form of cooling fins, usually made an integral part of the cylinder barrel and arranged either longitudinally along the barrel or circumferentially around it. The problem considered is that of reducing actual geometrical area of cooling surface, which is, of course, nonuniform in temperature, to equivalent "cooling" area at a definite, easily specified temperature. This may be done by finding an expression for the effectiveness of fin surface, i. e., the ratio of the amount of heat dissipated by unit area of fin surface to that dissipated by an equal area of cylinder wall surface with the same tempearture as that at the fin base. This will make it possible to treat all the cooling surface as if it were on the cylinder wall and had 100 per cent effectiveness.

II. APPROXIMATE FORMULA FOR EFFECTIVENESS OF COOLING FINS.

Two cases occur in practice: (1) Circumferential fins, usually of tapering thickness, with a base temperature that may change from point to point; (2) longitudinal fins with similar conditions of thickness and temperature. A direct analytical investigation of each case in all its generality is quite difficult and has not yet been completely worked out. In this paper an indirect attack on the general problem is made by a method of successive approximations. The effectiveness is first computed for a simple case, where several simplifying assumptions are made. Then the limitations of these assumptions are removed, one or two at a time, and the necessary correction made to the first result. The effectiveness f is, therefore, expressed in the form

$$f=f'+\Delta_1 f+\Delta_2 f+\cdot\cdot$$

GENERAL ASSUMPTIONS.

Four general assumptions of physical nature are made that apply to every case considered here:

(1) Quantity of heat transferred per unit time from the metal surface to the air is proportional to the temperature difference between the metal and the air.

(2) The coefficient q, heat transferred in unit time from a unit surface per unit temperature difference, is constant over the fin surface.

(3) The fin is symmetrical about a plane through its middle and approximately parallel to its faces.

(4) The temperature at a given point is independent of the time.

Assumption (1) is known as Newton's Law of Cooling. It has been found to be sensibly true for very high velocity air-stream cooling in such cases as are under consideration.[1]

Some preliminary measurements of heat dissipation by air-cooled engine cylinders made in the laboratories of the Bureau of Standards also indicate the validity of the assumption within limits necessary to work of this kind.

Assumption (3) is obviously true mechanically except for minor inequalities of manufacture, which on an average over a cylinder would be inappreciable. Physically, there is marked lack of symmetry about such a plane if the fin be oblique to the air stream, but this lack of symmetry has to do with assumption (2).

Assumption (4) provides a working basis that is entirely acceptable. Questions of engine design must be settled from considerations of cooling capacity under full load and continuous operation, when a steady temperature distribution exists. A cooling system which will meet this demand will obviously meet the less stringent cooling requirements involved in starting up and approaching temperature equilibrium. The rapid variations in temperature of the inside wall of the cylinder between explosion and intake are quite damped out in their effects on fin temperature. The validity of the assumption has been demonstrated by experimental evidence.[2]

Assumption (2) involves, amongst other things, either independence of the heat transferred from a fin to an air stream and the velocity of the air stream, or else the assumption of constant velocity of the air stream over all portions of the fin. The first hypothesis is untenable; the second one is discussed below. Experimental data are very incomplete. The assumption is recognized as weak, but in the present state of knowledge it is about all that can be done. It is known to be justifiable in long tubes and is probably not far wrong for longitudinal fins with fairly open spacing. One set of measurements[3] on a plain cylinder without fins, a small cylindrical rod (diameter 2 cm., $\frac{3}{4}$ inch) indicated that with the air stream normal to the axis of the cylinder, the variation in air velocity, front and rear, was of the order of 30 per cent, namely, a difference of 15 per cent each way from the mean value. British measurements on air velocity between fins indicate a change in q from tip to root of longitudinal fins of less than 15 per cent. These meager data would suggest that if an average value over a fin were taken, the deviations from it would not generally be more than 10 or 15 per cent. The error after integration in such cases is generally less than the original error, so that the result with this approximation is probably in the neighborhood of the true value.

SIMPLIFYING ASSUMPTIONS.

In addition, the following simplifying assumptions give a simple problem which serves as a basis for more complete analysis:

(1) The temperature across the fin thickness is sensibly constant.

(2) The fin is so long that the effect of the exposed ends (of longitudinal fins) is neglible.

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[1] A. H. Gibson. Automotive Industries, May 13, 1920, p. 1109. Theories and Practices in the Air Cooling of Engines.

[2] Judge, "High-Speed Internal Combustion Engines," pp. 107-109.

[3] T. E. Stanton, Great Britain Advisory Committee for Aeronautics, Report 94, 1912-13, p. 47.

(3) The heat loss from the exposed edge can be accounted for by imagining the width extended by a distance equal to one-half the fin thickness at the outer edge and assuming no heat loss from the end.

(4) The base temperature is constant.

(5) The fin thickness is constant.

(6) The fin is longitudinal.

The details of assumption (3) are illustrated in Figure 1.

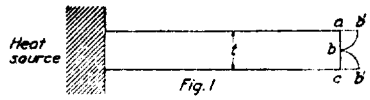


Fig. 1

The edge abc at a temperature Θ, dissipates some heat ΔH_1. If ab and bc be swung around to ab' and cb' and the space filled in with metal, the temperature drop from a to b' will be small and these surfaces will dissipate practically the same amount of heat as before. The other simplifying assumptions require no explanation.

It will be shown later that all of the Δf's are small compared with f', so that the cross products which represent the effects due to the joint action of several disturbing influences may be neglected and only first-order effects due to single causes need be worked out.

OUTLINE OF THE GENERAL PLAN OF MATHEMATICAL ANALYSIS.

First, f' is computed on the basis of all six simplifying assumptions stated above.

Second, f is computed with assumptions (1) and (3) removed. Thus $f-f'$ gives $\Delta_1 f+\Delta_3 f+\Delta_{1,3} f$. These are shown to be negligible for all conditions prevailing in engine work.

Third, f is computed with assumption (4) removed, giving $\Delta_4 f$, which is zero.

Fourth, f is computed with assumption (5) removed and replaced by the one that the sides are straight, and hence the thickness t is a linear function of the distance x from the fin tip. This gives $\Delta_5 f$, which is often not negligible. If λ is the ratio of the thickness at the tip to the average thickness, then $\Delta_5 f$ may be expressed in terms of f' and λ. Its value is computed for several values of λ and the results shown graphically on a chart.

Fifth, f is computed with assumption (6) removed and a longitudinal fin replaced by a circumferential one. This determines $\Delta_6 f$. If ρ is the ratio of the inner radius to the outer, f may be expressed in terms of f' and ρ only, so that the results for $\Delta_6 f$ are expressed graphically in the same way as above.

By means of these charts, f can be found with a sufficient degree of approximation for the most general case.

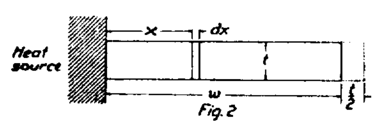


Fig. 2

(1) Computation of f'. (The approximate function to represent heat flow in terms of geometrical and thermal properties of the fin.)

Let q = coefficient of heat transfer from surface.

w = true width of fin.

t = fin thickness.

x = distance from the cylinder wall.

f = fin effectiveness.

k = fin conductivity.

θ = temperature of fin above the air.

θ_0 = temperature of cylinder wall above the air.

H = heat dissipated by the fin per unit time.

H_0 = heat dissipated by equal area of wall surface per unit time.

w' = corrected fin width $= w+\frac{t}{2}$.

By introducing the simplifying assumptions (3) and (4), the boundary conditions are:

$$\frac{d\theta}{dx}=0 \text{ when } x=w'$$
$$\theta=\theta_0 \text{ when } x=0 \tag{1}$$

The fundamental equation of heat transfer[4] under the conditions corresponding to assumptions (1), (2), and (5) is

$$\frac{d^2\theta}{dx^2}=\frac{2q\theta}{kt} \tag{2}$$

The solution of this equation in the form most convenient for this work is

$$\theta=A \cosh a\ (x-B) \tag{3}$$

where A and B are arbitrary constants of integration and a is an abbreviation for $\sqrt{2q/kt}$.

When A and B are determined to satisfy the boundary conditions, stated mathematically in (1)

$$\theta=\theta_0 \frac{\cosh a\ (x-w')}{\cosh aw'}$$

The rate of heat dissipation from a unit length of fin is computed by an integration with respect to x from 0 to w'.

$$H=2\int_0^{w'} q\ \theta\, dx=\frac{2q\theta_0}{\cosh aw'}\int_0^{w'} \cosh a\ (x-w')\, dx \tag{4}$$

The heat dissipation from an equal area ($2w'$) of cylinder wall at temperature θ_0 is

$$H_0=2q\theta_0 w'$$

and therefore

$$f'=\frac{H}{H_0}=\frac{1}{w' \cosh aw'}\int_0^{w'} \cosh a\ (x-w')\, dx=\frac{\tanh aw'}{aw'} \tag{5}$$

This function $(\tanh aw')/aw'$, for which the single letter f' will be used, is the function which will serve as the basis of discussion for much of the following work. Under average conditions of practice it will be found sufficiently exact to serve as the basis for all computations. In those cases where it does not fit with sufficient accuracy, it will be found convenient to use it as a principal term plus necessary correcting terms. The principal properties of the function therefore merit attention.

When aw' is increased, the value of $\tanh aw'$ increases likewise, but rather slowly, and although reaching 0.9 when $aw'=1.50$, it does not increase beyond 1.0, no matter how large aw' becomes. The ratio f', therefore, starts at unity and gradually decreases to zero, when plotted against values of aw'. This plot is shown in Figure 3, where the single letter u is substituted for aw'.

$$u=aw'=w'\sqrt{\frac{2q}{kt}} \tag{6}$$

[4] The method of deriving this equation is explained fully in elementary textbooks on heat. The difference in quantity of heat conducted into an elemental slab at coordinate x and that conducted out at $x+dx$ is, in the equilibrium state, the amount that escapes into the air through the surface dx. This equality, when common factors are canceled, is the equation (2).

Figure 3 shows that f grows less as u increases, whence as w' or q increase, or as k or t decrease. That is to say, the effectiveness of a fin decreases as—

(1) The fin width is made greater.

(2) The fin is made thinner.

(3) The fin is made of material with a poorer thermal conductivity.

(4) The coefficient of heat dissipation to the air increases.

This last statement indicates that fin effectiveness is relatively less at higher air speeds, since the value of q increases with air speed.

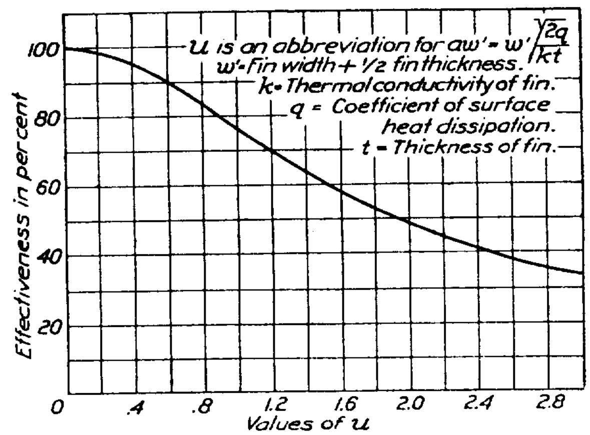


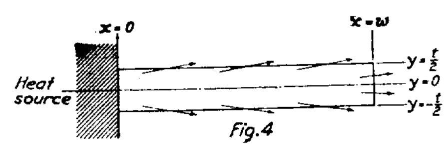
Fig. 3.—Curve showing function $\frac{\tanh u}{u}$. Any units whatever if they belong to a mutually consistent system will do to measure the above four quantities and will lead to the same number for U.

III. CORRECTION TERMS TO APPROXIMATE FORMULA.

In the use of the formula (5) derived in the preceding section, it is very necessary to know the departure from exactness which has been introduced by use of the simplifying assumptions which are obvious deviations, at least to a small degree, from the conditions which actually prevail. This investigation of the order of magnitude of the several correction terms involves considerable tedious mathematics which are incorporated in the paper only to meet the needs of a specialist in the field. The general reader will find the conclusions summarized at the end of the chapter.

(1) CORRECTIONS FOR HEAT DISSIPATION FROM EDGE OF FIN AND FOR "CROSS-FLOW" IN FIN.

This correction is determined by solving the problem as stated initially without the aid of simplifying assumptions (1) and (3) but with the aid of the remaining simplifications. The difference between the solution so obtained and the function defined in equation (5) above shows the value of the correction which would be necessary to the latter to take account of the two factors of this title.



The problem is a straightforward development of two-dimensional heat flow[5], in which the boundary conditions are stated mathematically as follows:

With the origin of coordinates located as indicated in Figure 4, the axis $y=0$ is the median line of the fin, and from considerations of symmetry there can be no heat flow across the median plane, a condition expressed mathematically by zero temperature gradient.

[5] Byerly-Fouriers Series and Spherical Harmonics, art. 59, p. 102, edition of 1902.

$$\text{Along } y=0 \quad \frac{\partial \theta}{\partial y}=0 \tag{7}$$

$$\text{Along } y=t/2 \quad k\frac{\partial \theta}{\partial y}=-q\theta \tag{8}$$

(this face being one which is dissipating heat to the air).

$$\text{Along } x=w \quad k\frac{\partial \theta}{\partial x}=-q\theta \tag{9}$$

$$\text{Along } x=0 \quad \theta=\theta_0 \tag{10}$$

(this edge of the fin being the one integral with the cylinder wall and maintained at given constant temperature θ_0). The fundamental equation of heat transfer for this problem is

$$\frac{\partial^2 \theta}{\partial x^2}+\frac{\partial^2 \theta}{\partial y^2}=0 \tag{11}$$

which, being of the second order in two variables, can include in its primitive four arbitrary constants and permit of applying the four conditions (7) to (10) for the determination of such constants to give a complete solution of the problem as stated.

The convenient form for the primitive is

$$\theta=\sum_{1}^{\infty}\text{m}\, A_m \cosh \alpha\, (x-B) \cos \alpha y \tag{12}$$

which contains three arbitrary constants A, B, α and satisfies the condition (7). The development of condition (8) leads to determination of α as any solution of

$$\frac{\alpha t}{2} \tan \frac{\alpha t}{2}=\frac{qt}{2k} \tag{13}$$

Then the value of B is defined in terms of α by the use of condition (9) leading to

$$\alpha \tanh \alpha (B-w)=\frac{q}{k} \tag{14}$$

Lastly, values of A_m must be selected to conform to condition (10) in terms of the α and B already specified. These values of A must satisfy

$$\theta_0=\sum_{1}^{\infty}\text{m}\, A_m \cosh \alpha B \cos \alpha y \tag{15}$$

The possibility of determining values of A_m to satisfy this relation has been established by workers in the field of Fouriers Series and other harmonic expansions,[6] and while the particular form here applied may not identify exactly with those commonly found in the textbooks, it seems quite unnecessary to supply here any of the intervening transformations. It is adequate evidence of the validity of the assumption that definite values for A_m may be found if we proceed to find some which define a convergent series.

The principal reductions will have to do with equation (13) and the algebra is simplified considerably by the use of a parameter ϕ in place of α, defined by the relation

$$\phi=\frac{\alpha t}{2} \tag{16}$$

[6] Byerly-Fouriers Series and Spherical Harmonics, pp. 118-121.

It may be noted that equation (13) is one with an infinite number of real values of α or ϕ satisfying it, whereas equation (14) has but one value of B for any given value of α, the hyperbolic tangent being a single valued function. To dispose of this equation and replace it by a series, the quantity B is replaced by a new parameter ϵ, defined by

$$B=w+\frac{t}{2}+\epsilon \tag{17}$$

In terms of ϵ and ϕ the revised equations (13) and (14) take the form

$$\phi \tan \phi=\frac{qt}{2k} \tag{18}$$

$$\phi\left(1+\frac{2\epsilon}{t}\right)=\tanh^{-1}\frac{qt}{2k\phi}=\frac{qt}{2k\phi}+\frac{1}{3}\left(\frac{qt}{2k\phi}\right)^3+\frac{1}{5}\left(\frac{qt}{2k\phi}\right)^5+\cdots$$

$$\epsilon=\frac{-t}{2}+\frac{k}{q}\left(\frac{qt}{2k\phi}\right)^2\left[1+\frac{1}{3}\left(\frac{qt}{2k\phi}\right)^2+\frac{1}{5}\left(\frac{qt}{2k\phi}\right)^4+\cdots\right] \tag{19}$$

With a ϕ_m determined as a root of equation (18) and the corresponding ϵ_m determined from the series (19), we must select an A_m to give the expansion (15) which in the new parameters is

$$\theta_0=\sum_1^\infty{}^m A_m \cosh\frac{2\phi_m}{t}\left(w+\frac{t}{2}+\epsilon_m\right)\cos\frac{2\phi_m}{t}y \tag{20}$$

and then substitute all these in (12) revised to

$$\theta=\sum_1^\infty{}^m A_m \cosh\frac{2\phi_m}{t}\left(x-w-\frac{t}{2}-\epsilon_m\right)\cos\frac{2\phi_m}{t}y \tag{21}$$

The details pertaining to the process just outlined are tedious and of interest only to the worker who desires to check the development. Multiply each side of (20) by $\cos(2\phi_k y/t)\,dy$, where ϕ_k is any root of equation (18) and integrate for y from 0 to $t/2$.

$$\int_0^{t/2}\theta_0\cos\frac{2\phi_k}{t}y\,dy=\sum_1^\infty{}^m A_m\cosh\frac{2\phi_m}{t}\left(w+\frac{t}{2}+\epsilon_m\right)\int_0^{t/2}\cos\frac{2\phi_m}{t}y\cos\frac{2\phi_k}{t}y\,dy \tag{22}$$

$$\int_0^{t/2}\cos\frac{2\phi_m}{t}y\cos\frac{2\phi_k}{t}y\,dy=\frac{t}{2}\left[\frac{\phi_m\sin\phi_m\cos\phi_k-\phi_k\sin\phi_k\cos\phi_m}{\phi_m^2-\phi_k^2}\right]$$

and therefore vanishes for those values of ϕ_m and ϕ_k which are roots of equation (18) except for the particular choice $\phi_k=\phi_m$, where the indeterminate form of the expression introduces the possibility of finite value. It follows, therefore, that the summation on the right-hand side of equation (22) can consist of no more than the single term given by the equality of m and k, whence

$$\int_0^{t/2}\theta_0\cos\frac{2\phi_m}{t}y\,dy=A_m\cosh\frac{2\phi_m}{t}\left(w+\frac{t}{2}+\epsilon_m\right)\int_0^{t/2}\cos^2\frac{2\phi_m}{t}y\,dy$$

$$=A_m\cosh\left[\frac{2\phi_m}{t}\left(w+\frac{t}{2}+\epsilon_m\right)\right]\times\frac{t}{4}\left[1+\frac{\sin 2\phi_m}{2\phi_m}\right]$$

From which

$$A_m=\frac{4}{t}\,\frac{\operatorname{sech}\frac{2\phi_m}{t}\left(w+\frac{t}{2}+\epsilon_m\right)\int_0^{t/2}\theta_0\cos\frac{2\phi_m}{t}y\,dy}{1+\frac{\sin 2\phi_m}{2\phi_m}} \tag{23}$$

The above value of A_m is in a form for a general expansion of θ_0 as any arbitrary function of y, namely, for any specified temperature along the thickness edge of the fin (i. e., along the cylinder wall). This generality is too complicated for consideration in this paper, which has been limited to the case of θ_0 being constant. This condition may therefore be introduced at this point into equation (23), which reduces to

$$A_m = 4\theta_0 \operatorname{sech} \frac{2\phi_m}{t} (w + t/2 + \epsilon_m) \frac{\sin \phi_m}{2\phi_m + \sin 2\phi_m} \tag{24}$$

This value of A_m used in equation (21), with the aid of equations (18) and (19), completes the solution of the problem as a problem in thermal conduction.

For application to the purposes of this paper, it is desired to compute the effectiveness of the fin surface, which has been defined above as the ratio of the heat dissipated by the fin to the heat which would be dissipated in the same time by an equal area of surface all maintained at the constant, uniform temperature θ_0, which is here the temperature of one edge of the fin. Designate this ratio as H/H_0, and retain the previous notation with q for coefficient of surface heat dissipation, remembering that θ is measured on a temperature scale with its zero chosen at the temperature of the air stream cooling the fin.

$$H_0 = q\ (2w+t)\ \theta_0 \qquad H = q\int \theta dS$$

$$f = \frac{H}{H_0} = \frac{1}{2\left(w+\frac{t}{2}\right)\theta_0} \int \theta dS$$

To integrate over the dissipating surface, we have $dS = dx$ along the two fin surfaces for unit length of fin and $dS = dy$ along the edge opposite the cylinder wall.

$$\int \theta dS = 2\int_0^w \theta_{y=t/2}\, dx + 2\int_0^{t/2} \theta_{x=w}\, dy$$

$$f = \frac{\int_0^w \theta_{y=t/2}\, dx + \int_0^{t/2} \theta_{x=w}\, dy}{\theta_0\left(w+\frac{t}{2}\right)} \tag{25}$$

From equation (21)

$$\theta_{y=t/2} = \sum_1^\infty{}_m A_m \cosh \frac{2\phi_m}{t}\left(x - w - \frac{t}{2} - \epsilon_m\right) \cos \phi_m$$

$$\theta_{x=w} = \sum_1^\infty{}_m A_m \cosh \frac{2\phi_m}{t}\left(\frac{t}{2} + \epsilon_m\right) \cos \frac{2\phi_m}{t} y$$

and by substituting these values in equation (25) and reducing the result,

$$f = \frac{t}{w+\frac{t}{2}} \sum_1^\infty{}_m \frac{\sin 2\phi_m}{\phi_m\ (2\phi_m + \sin 2\phi_m)} \left[\tanh \frac{2\phi_m}{t}\left(w + \frac{t}{2} + \epsilon_m\right) - \frac{\sinh \frac{2\phi_m}{t}\left(\frac{t}{2} + \epsilon_m\right)}{\cosh \frac{2\phi_m}{t}\left(w + \frac{t}{2} + \epsilon_m\right)} \right]$$

$$+ \frac{t}{w+\frac{t}{2}} \sum_1^\infty{}_m \left[\frac{2 \sin^2 \phi_m}{\phi_m\ (2\phi_m + \sin 2\phi_m)} \frac{\cosh \frac{2\phi_m}{t}\left(\frac{t}{2} + \epsilon_m\right)}{\cosh \frac{2\phi_m}{t}\left(w + \frac{t}{2} + \epsilon_m\right)} \right] \tag{26}$$

In the use of equation (26) the terms of the summation are given by using for ϕ_m in succession all the real roots of equation (18)

$$\phi \tan \phi = \frac{qt}{2k} \tag{18}$$

q = heat dissipated from fin to air stream per unit time per unit area of fin surface per unit temperature difference.

t = fin thickness.

k = thermal conductivity of fin material.

From each successive value of ϕ_m, the corresponding ϵ_m is computed from equation (19):

$$\epsilon_m = \frac{-t}{2} + \frac{k}{q}\left(\frac{qt}{2k\phi_m}\right)^2\left[1+\frac{1}{3}\left(\frac{qt}{2k\phi_m}\right)^2 + \ . \ . \ .\right] \tag{19}$$

It is obvious that equation (26) is much too cumbersome for everyday application. From the standpoint of pure mathematics it is the equation to choose in preference to equation (5), being based upon more acceptable assumptions and including as a special case the conditions leading to the solution expressed by equation (5). From the mathematical viewpoint, correct procedure would be to solve the problem in the more general form, obtaining equation (26) for effectiveness of a fin, and then to examine the relative magnitude of the terms involved and leave out the small ones in order as successively less exact approximations are desired. In this way one might approach a comparatively simple formula, either equation (5) or an equivalent, for all usual applications. In this paper an alternative presentation has been adopted in deference to the algebraic complexity of the processes in the more general case, and it has been deemed wise to develop first the comparatively simple solution which is good enough to apply to most air-cooled engine cylinder work, showing how good or how bad is the approximation by a later development of the more exact relations.

These more exact relations are too complicated for convenience. The course outlined above can be put in practice, not by a general consideration of the magnitude of each term in the series, but only by specific numerical computation of these terms for one or more typical sets of conditions.

The units selected for q, k, and t are immaterial so long as they are consistent. The factor qt/k, which enters the computation, is dimensionless and independent of the unit system. If it be desired to measure t in inches and q in Btu. per minute per square inch per degree Fahrenheit, then it is only necessary to express k, the thermal conductivity, in Btu. per minute per square inch per unit temperature gradient in degrees Fahrenheit per linear inch. The international units are used below. Assume:

$k=0.10$ calories per second per cm.^{2} per unit gradient in °C. per cm.).

$q=0.008$ (calories per second per cm.^{2} per °C.).

$t=0.5$ cm.

$w=3.0$ cm.

This set of values is for a steel fin ($k=0.10$) of excessive proportionate thickness (one-sixth of the width) and large absolute dimensions, and in a very high speed wind stream. ($q=0.008$ probably corresponds to a wind velocity of 90 meters per second, 200 miles per hour.) These conditions are the ones which should exaggerate the effects of "cross flow" and dissipation from the thin edge of the fin, the two factors included in the more complicated solution and omitted in the simpler. By "cross flow" is meant taking into account the two-dimensional flow of heat in a plane section of the fin rather than treating it as linear flow from the engine cylinder wall toward the outer edge of the fin, sensibly parallel to the flat fin surfaces and with but a negligible component perpendicular to these surfaces or across the fin. The case selected is therefore unfavorable to the approximate equation and should indicate the largest corrections to it necessary in any ordinary application.

$$\frac{qt}{2k} = \frac{0.008 \times 0.5}{0.20} = 0.020 \tag{27}$$

The values of ϕ_m are given by the roots of—

$$\phi \tan \phi = \frac{1}{50} \text{ or } \tan \phi = \frac{1}{50\phi} \tag{28}$$

which is conveniently solved by graphical means by plotting values of $\tan \phi$ and of $1/50\phi$ and reading the value of ϕ at the intersections, namely, where the two functions equal each other (fig. 5). The value of $1/50\phi$ are so small for all values of ϕ greater than that corresponding to the first few roots that the function may be taken as coincident with the axis and cutting the tangent curves at $m\pi$. The first root may be read from the graph of figure 5 plotted to a scale sufficiently open or may be approximated analytically by the series expansion,

$$\tan\phi=\phi+\frac{\phi^3}{3}+\frac{2\phi^5}{15}+\cdot\cdot\cdot\text{ if }\phi<\frac{\pi}{2} \qquad 0.02=\phi^2+\frac{\phi^4}{3}+\frac{2\phi^6}{15}$$

(neglecting high powers of ϕ because it is noted that ϕ is much smaller than unity),

$$0.1414=\phi\left[1+\frac{\phi^2}{3}+\frac{2\phi^4}{15}\right]^{\frac{1}{2}}$$

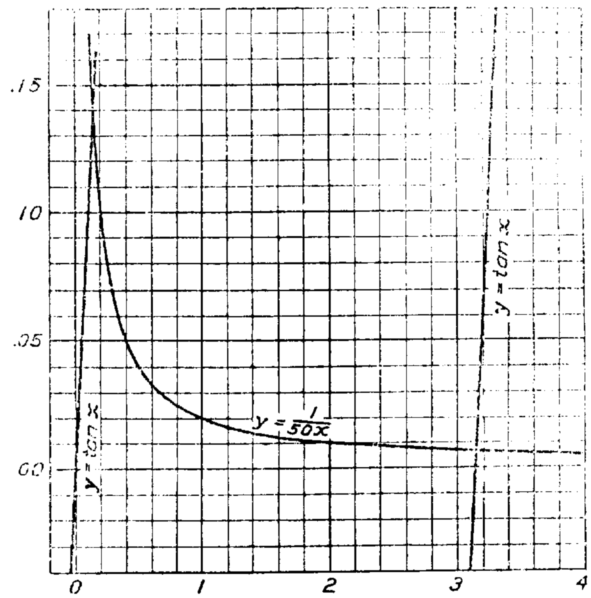


FIG. 5. Graphic solution of $\tan x=\frac{1}{50x}$

To solve this it is evident that ϕ is so closely 0.1414 that the latter may be substituted for ϕ^2 and ϕ^4 without appreciable error in comparison with unity.

$$\begin{aligned}0.1414&=\phi[1+0.00667+0.00005]^{\frac{1}{2}}\\&=\phi[1+0.00336]\\\phi&=0.1414\times[1-0.00336]=0.1409\end{aligned}$$

To find the second root, ϕ_2, the relation $\tan(\pi+\alpha)=\tan\alpha$ is the key to the process, bearing in mind that ϕ_2 is so very close to π as to admit of some approximations which are very exact while at the same time very simple. Omitting details

$$\phi_2=\pi+0.00635$$

The departure of ϕ_3 from 2π and of ϕ_4 from 3π, etc., is quite inappreciable, whence the values of ϕ for equation (26) are:

$$\phi_1=0.1409 \quad \phi_2=3.148 \quad \phi_3=6.28 \quad \phi_4=9.42,\text{ etc.}$$

The next step in computation is evaluation of ϵ_m from equation (19).

$$\frac{qt}{2k}=0.020 \qquad \frac{k}{q}=\frac{0.100}{0.008}=12.50$$

$$\frac{qt}{2k\phi_1}=\frac{0.020}{0.1409}=0.1419 \qquad \frac{qt}{2k\phi_2}=\frac{0.020}{3.148}=0.0063 \qquad \frac{qt}{2k\phi_3}=0.0032$$

$$\epsilon_1+\frac{t}{2}=12.50\ (0.1419)^2\left[1+\frac{1}{3}\ (.1419)^2+\frac{1}{5}\ (0.1419)^4+\cdot\cdot\cdot\right]=0.25356$$

$$\epsilon_2+\frac{t}{2}=12.50\ (0.0063)^2\left[1+\frac{1}{3}\ (0.0063)^2+\cdot\cdot\cdot\right]=0.00049$$

$$\epsilon_3+\frac{t}{2}=12.50\ (0.0032)^2\left[1+\frac{1}{3}\ (0.0032)^2+\cdot\cdot\cdot\right]=0.00013$$

In substituting the values just obtained into the terms of equation (26) it is worth while to tabulate the intermediate steps for the first term ($m=1$) in order to illustrate the order of magnitude of the different factors and to form a guide for estimation of the changes to be expected in using somewhat different values for k or q or t from those selected for this example.

$$\phi_1=0.1409 \qquad \epsilon_1+\frac{t}{2}=0.25356 \qquad w+\frac{t}{2}+\epsilon_1=3.2536$$

$$2\phi_1=0.2818 \qquad \frac{2\phi_1}{t}\left(\frac{t}{2}+\epsilon_1\right)=0.1429 \qquad \frac{2\phi_1}{t}\left(w+\frac{t}{2}+\epsilon_1\right)=1.834$$

$$\sin 2\phi_1=0.2780 \quad \sinh\frac{2\phi_1}{t}\left(\frac{t}{2}+\epsilon_1\right)=0.1434 \qquad \tanh\frac{2\phi_1}{t}\left(w+\frac{t}{2}+\epsilon_1\right)=0.950$$

$$\sin^2\phi_1=0.0197 \quad \cosh\frac{2\phi_1}{t}\left(\frac{t}{2}+\epsilon_1\right)=1.0102 \qquad \cosh\frac{2\phi_1}{t}\left(w+\frac{t}{2}+\epsilon_1\right)=3.21$$

$$f=\frac{t}{w+\frac{t}{2}}\left\{\frac{0.2780}{0.1409\ (0.5598)}\left[0.950-\frac{0.1434}{3.21}\right]+\frac{0.0394}{0.1409\ (0.5598)}\ \frac{1.0102}{3.21}\right\}$$

$$+\frac{t}{w+\frac{t}{2}}\sum_2^\infty m\frac{\sin 2\phi_m}{\phi_m\ (2\phi_m+\sin 2\phi_m)}\left[\tanh\frac{2\phi_m}{t}\left(w+\frac{t}{2}+\epsilon_m\right)-\frac{\sinh\frac{2\phi_m}{t}\left(\frac{t}{2}+\epsilon_m\right)}{\cosh\frac{2\phi_m}{t}\left(w+\frac{t}{2}+\epsilon_m\right)}\right]$$

$$+\frac{t}{w+\frac{t}{2}}\sum_2^\infty m\frac{2\sin^2\phi_m}{\phi_m\ (2\phi_m+\sin 2\phi_m)}\ \frac{\cosh\frac{2\phi_m}{t}\left(\frac{t}{2}+\epsilon_m\right)}{\cosh\frac{2\phi_m}{t}\left(w+\frac{t}{2}+\epsilon_m\right)} \tag{29}$$

$$f=0.1538\{3.524\ [0.950-0.0447]+0.4995\times 0.315\}+\Sigma,\text{ etc.}$$

$$=0.1538\{3.189+0.1572\}+\Sigma,\text{ etc.}$$

$$=0.5148+0.1538\ \frac{\sin\ (2\pi+0.0127)}{3.148\ (6.30+0.013)}\ \tanh\ 12.59\ (3.0005)$$

$$-\left.\frac{\sinh\ 12.59\ (0.00049)}{\cosh\ 12.59\ (3.0005)}\right]+\frac{2\sin^2\ (\pi+0.00635)}{3.148\times 6.31}\ \left.\frac{\cosh\ 12.59\ (0.00049)}{\cosh\ 12.59\ (3.0005)}\right\}+\sum_{m=3},\text{ etc.} \tag{30}$$

Reduction of the terms in the above expression (30) indicates by inspection the general trend of each term in the later series. Cosh 12.59 × 3 is enormous and the two cosh terms in the denominators above and in all succeeding terms are so extremely large with respect to anything occurring in the numerators that the terms containing them are inappreciable. Also, the tanh function of a large argument is unity, whence there is left, to evaluate, only

$$\frac{\sin\ (2\pi+\ 0.0127)}{3.148\ (6.30+0.013)}\ \text{and}\ \sum_3^\infty m\frac{\sin 2\phi_m}{\phi_m\ (2\phi_m+\ \sin 2\phi_m)}$$

where ϕ_m for $m=3$ and greater is so near a multiple of π that $\sin 2\phi_m$ will be inappreciable. The numerical term above is

$$\frac{0.0127}{3.15\times 6.31}=0.00064$$

and

$$f=0.5148+0.1538\times 0.0006=0.5149 \tag{31}$$

This value of f should now be contrasted with the approximate value f' given by the simple expression, equation (5):

$$f'=\frac{\tanh aw'}{aw'} \qquad a=\sqrt{\frac{2q}{kt}} \qquad w'=w+\frac{t}{2}$$

$$q=0.008$$
$$k=0.10$$
$$t=0.50$$
$$w=3.00$$
$$w'=3.25$$
$$a=\sqrt{\frac{0.016}{0.050}}=\sqrt{0.32}=0.565$$
$$aw'=1.836$$
$$\tanh aw'=0.950$$
$$f'=\frac{0.950}{1.836}=0.5176 \tag{32}$$

The difference between f' and the more exact value of the effectiveness which is given by f in equation (31) is therefore not quite 3 parts in 500, or less than 0.6 per cent.

It thus appears that the error introduced by neglecting the transverse temperature gradient and assuming the edge correction to be simply $t/2$ added to w is less than 1 per cent in this exaggerated instance. In ordinary cases it is negligible entirely, for an aluminum fin, $q=0.003$ (air speed 70 or 80 mi./hr.), $k=0.50$, $w'=2.0$ cm, $t=0.15$ cm, $f=f'$, within less than 0.1 per cent.

(2) CORRECTIONS FOR VARYING BASE TEMPERATURE AND EXPOSED ENDS.

In this proof the edge correction that has been proved to be sufficient, i. e., $w+t/2$ for w and $l+t/2+t/2$ for the length, will be assumed and the transverse temperature gradient neglected. Hence, the problem becomes that illustrated in Figure 6.

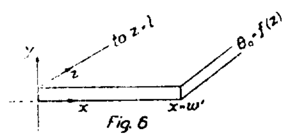


Fig. 6

The plate of width w is replaced by the fictitious one of width w' with the origin at the outer or free edge, so that the plane $x=w'$ becomes the engine cylinder wall maintained at a given temperature θ_0, assumed constant as to time but not uniform with relation to the coordinate z. By neglecting the transverse (y direction) temperature gradient, the fundamental equation for the problem takes the form

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial z^2}=\frac{2q}{kt}\theta \tag{33}$$

which must satisfy boundary conditions at the four edges as follows:

when

$$z=0 \qquad \frac{\partial\theta}{\partial z}=0$$

$$z=l'=l+t \qquad \frac{\partial\theta}{\partial z}=0$$

$$x=0 \qquad \frac{\partial\theta}{\partial x}=0$$

$$x=w' \qquad \theta=\theta_0=F(z)$$

where $F(z)$ is a prescribed function, given for any particular problem. The boundary conditions for the two flat surfaces of the fin, which are cooled by the air stream, have been incorporated into equation (33) in the process of deducing it from fundamental considerations.

A convenient form of solution of equation (33) for this application is

$$\theta = \Sigma A \cosh \alpha x \cos \beta z \tag{34}$$

where α and β are connected by the relation

$$\alpha^2 - \beta^2 = \frac{2q}{kt} \tag{35}$$

because in the form selected for (34) the boundary conditions for $x=0$ and $z=0$ are automatically satisfied for all values of A and α, leaving these two parameters to be determined by the remaining two conditions. The requirement at $z=l'$ is satisfied if

$$\beta = \frac{m\pi}{l'} \tag{36}$$

where m is any integer whatever. The solution is complete in the form

$$\theta = \sum_0^\infty {}^{m} A_m \cosh\left(\sqrt{\frac{2q}{kt} + \frac{m^2\pi^2}{l'^2}}\, x\right) \cos \frac{m\pi z}{l'} \tag{37}$$

provided values of A_m are determinable, so that when $x=w'$ the function (37) identifies with the given $F(z)$. This is a common Fouriers Series development and requires for A_m the value

$$A_m = \frac{2}{l' \cosh \sqrt{\frac{2q}{kt} + \frac{m^2\pi^2}{l'^2}}\, w'} \int_0^{l'} F(z) \cos \frac{m\pi z}{l'} dz \tag{38}$$

It is not worth while for purposes of this paper to assume any of the more probable forms for $F(z)$ and complete the solution of such special cases. It happens that the general conclusion which is desired, namely, the difference between the approximate value of fin effectiveness which is given by equation (5) and the more exact value given by equation (38), is capable of being found in terms of a general form for $F(z)$ unreduced to special cases.

To compute the effectiveness of the fin, proceed according to the definition to find expressions for the heat actually dissipated and that which would be dissipated if each portion of the fin were at the temperature of its contiguous cylinder wall. The latter quantity of heat is:

$$H_0 = 2w'q \int_0^{l'} F(z)\, dz \tag{39}$$

and the actual dissipation is

$$H = 2q \int_0^{w'} dx \int_0^{l'} \theta dz \tag{40}$$

From the ratio of these, the effectiveness f is

$$f = \frac{H}{H_0} = \frac{\int_0^{w'} \int_0^{l'} \theta dx\, dz}{w' \int_0^{l'} F(z)\, dz} \tag{41}$$

When the value of θ defined by equation (37) is substituted in (41), it contains the expression

$$\int_0^{l'} \cos \frac{m\pi z}{l'}\, dz$$

which is zero except for the value $m=0$, and, accordingly, the working expression for equation (41) is much simplified. The zero term of a cosine Fourier expansion is half the formula for the general term; that is to say, equation (38) reduces to

$$A_0=\frac{1}{2}\frac{2}{l'\cosh\sqrt{\frac{2q}{kt}}w'}\int_0^{l'}F(z)\,dz$$

and

$$f=\int_0^{w'}\int_0^{l'}\frac{A_0\cosh\sqrt{\frac{2q}{kt}}x\,dx\,dz}{w'\int_0^{l'}F(z)\,dz}=\frac{l'\sqrt{\frac{kt}{2q}}A_0\sinh\sqrt{\frac{2q}{kt}}w'}{w'\int_0^{l'}F(z)\,dz} \tag{42}$$

The substitution in equation (42) of the expanded form of A_0 reduces the result to the form

$$f=\frac{\tanh\sqrt{\frac{2q}{kt}}w'}{\sqrt{\frac{2q}{kt}}w'} \tag{43}$$

which expression is identical with the value given by equation (5), derived on the hypothesis that the fin base did not vary in temperature along its length. In other words, the nature of the variation of temperature along the base of the fin, including a uniform distribution as a special case, is immaterial, in so far as the function to express fin effectiveness is concerned.

(3) WEDGE-SHAPED FINS.

The complete solution for a wedge-shaped fin involves rather complicated mathematics, the terms involved being Bessel functions with imaginary arguments. The practical form of application of the solution, expressed in manageable form for numerical work, is to plot a curve or series of curves giving the correction to be applied to a simple expression for the effectiveness of the nearest equivalent parallel-sided fin. A digest of the mathematical treatment follows.

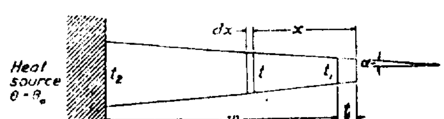


Fig. 7

Let the fin be wedge-shaped, as shown in Figure 7, with straight sides, the thickness tapering from a value t_2 at the fin base to a value t_1 at the fin tip. The fin width may be considered extended in amount $t_1/2$ to account for the heat dissipation actually occurring from the surface at the end (t_1) and the fictitious end is then treated as if blanketed completely. The origin of coordinates is most conveniently located at the fictitious end and the problem is thus stated in terms of surface dissipation along the two sloping surfaces of Figure 7, a blanket at $x=0$; namely, $d\theta/dx=0$ in that plane and $\theta=\theta_0$ at $x=w+t_1/2$. The fin is to be considered so long (direction perpendicular to plane of the paper) that the end dissipation is immaterial.

For derivation of the fundamental equation, consider the heat flow per unit time in a section dx at the point x, the corresponding fin thickness (t) being defined in terms of the angular parameter α shown in Figure 7.

$$t=t_1+2\left(x-\frac{t_1}{2}\right)\tan\alpha. \tag{44}$$

$$\frac{\partial t}{\partial x}=2\tan\alpha,$$

The heat flow, per unit time per unit length of fin, at the plane x is

$$kt\frac{\partial\theta}{\partial x}$$

(where k is thermal conductivity of the fin, θ the temperature at point x and x and t as defined in fig. 7) so that the difference between the heat flow into an elementary slice at plane x and out at plane $x + dx$ is

$$\frac{\partial}{\partial x}\left(kt\frac{\partial\theta}{\partial x}\right)dx, \text{ namely, } kt\frac{\partial^2\theta}{\partial x^2}dx+k\frac{\partial\theta}{\partial x}\frac{\partial t}{\partial x}$$

By means of equation (44) this expression becomes

$$kt\frac{\partial^2\theta}{\partial x^2}dx+2\tan\alpha\cdot k\frac{\partial\theta}{\partial x}dx \tag{45}$$

In the equilibrium condition, this quantity of heat equals the heat dissipated by the two elements of fin surface, namely,

$$2q\theta\frac{dx}{\cos\alpha}$$

(q = heat dissipation per unit area per unit time per unit temperature difference, fin surface to surrounding air; θ = temperature of fin at point x, as above, the scale of temperature used being such as to have zero for the temperature of the air)

$$k\left(t_1+\left[2x-\frac{t_1}{2}\right]\tan\alpha\right)\frac{\partial^2\theta}{\partial x^2}+2k\tan\alpha\frac{\partial\theta}{\partial x}=\frac{2q\theta}{\cos\alpha} \tag{46}$$

This equation is not one for which a solution may be recognized readily, the term causing trouble being the x in the coefficient of the second derivative. By a change of independent variable, involving considerable algebra, the equation reappears in a well-known form, similar to Fourier's equation, Bessel's equation of zero order. The clue to the necessary change of variable is furnished by examining equation (46) in the standard form with unity for the initial coefficient,

$$\frac{d^2\theta}{dx^2}+\frac{2\tan\alpha}{t_1(1-\tan\alpha)+2x\tan\alpha}\frac{d\theta}{dx}=\frac{2q\theta}{k\cos\alpha[t_1(1-\tan\alpha)+2x\tan\alpha]} \tag{47}$$

and trying substitutions that will simplify the coefficient of $\frac{d\theta}{dx}$. (It may be noted at this point that since in equation (46) θ has been expressed as a function of only one variable, x, it is unnecessary to distinguish further between partial and total derivatives.) The change of variable is defined by

$$\mu^2=4b^2\left[x+\frac{t_1(1-\tan\alpha)}{2\tan\alpha}\right] \tag{48}$$

where

$$b^2=\frac{q}{k\sin\alpha}$$

and the equation resulting from the transformation of (47) is

$$\frac{d^2\theta}{d\mu^2}+\frac{1}{\mu}\frac{d\theta}{d\mu}-\theta=0 \tag{49}$$

which would be Fourier's equation were the terms all positive. Upon substituting $(i\mu)$ for μ, the equation goes into this form, whence the solution of (49), as given, is

$$\theta=AJ_0(i\mu)+BK_0(i\mu) \tag{50}$$

where A and B are arbitrary constants and J_0 and K_0 are Bessel's functions of the two kinds, order zero. Before discussing briefly any properties of such functions relevant to this application, it is well to examine the more finished form taken by the equation when A and B are determined to satisfy the terminal conditions mentioned above,

$$\frac{d\theta}{dx}=0 \text{ when } x=0 \text{ and } \theta=\theta_0 \text{ when } x=w' \tag{51}$$

These do not correspond to simple algebraic expressions when expressed in the variable μ.

$$\frac{d\theta}{d\mu}=\frac{\mu}{2b^2}\frac{d\theta}{dx}$$

so that for all finite values of μ the derivative $d\theta/d\mu$ vanishes whenever $d\theta/dx$ vanishes, whence

$$\left.\begin{aligned}\frac{d\theta}{d\mu}=0 \text{ when } \mu=\mu_1=2b\sqrt{\frac{t_1\,(1-\tan\alpha)}{2\tan\alpha}}\\ \theta=\theta_0 \text{ when } \mu=\mu_2=2b\sqrt{w'+\frac{t_1\,(1-\tan\alpha)}{2\tan\alpha}}\end{aligned}\right\} \tag{52}$$

In order to apply these conditions to the determination of A and B in equation (50), it is necessary to make use of the properties of Bessel functions which have been discovered by mathematicians working in the field. Even a brief review of such properties is far beyond the scope of this paper.[7]

The Bessel function of the second kind, designated above as the K function, is, for a complex variable, generally replaced by a slightly different form of function than that for which the symbol K is common in mathematical literature. The common use of K makes it the function related to J, so that

$$K_0\,(x)=\log x\,J_0\,(x)+\frac{x^2}{2^2}-\frac{x^4}{(2\cdot 4)^2}\left(1+\frac{1}{2}\right)+\frac{x^6}{(2\cdot 4\cdot 6)^2}\left(1+\frac{1}{2}+\frac{1}{3}\right)-\ .\ .\ .$$

Accordingly, with an imaginary argument (ix), a complex relation would result,

$$K_0\,(ix)=\log i\,J_0\,(ix)+\log x\,J_0\,(ix)-\frac{x^2}{2^2}-\frac{x^4}{(2\cdot 4)^2}\left(1+\frac{1}{2}\right)-\ .\ .\ .$$

and it is convenient to take the term in $\log i$ over on the left-hand side of the above equation and define a new function which it will be noted is a real function in x. It is beyond the scope of this paper to consider in any detail the properties of such functions and the reasons for selecting particular forms as the elements in which to express solutions. Unfortunately, there is considerable difficulty in comprehending the literature on the subject because great confusion occurs in the notation employed by different writers. The original extensive treatment of complex Bessel functions is due to Hankel[8] and the symbol H is common for such functions but with exasperating lack of unanimity in regard to the exact definitions of such functions, which in the hands of various writers differ by several additive constants or constant multipliers. For the purposes of this paper, the H function employed will be that tabulated by Jahnke and Emde,[9] defined by the following series:

$$iH_0\,(ix)=\frac{2}{\pi}\left\{J_0\,(ix)\,\log\frac{2}{\gamma x}+\frac{x^2}{2^2}+\left(1+\frac{1}{2}\right)\frac{x^4}{(2\cdot 4)^2}+\left(1+\frac{1}{2}+\frac{1}{3}\right)\frac{x^6}{(2\cdot 4\cdot 6)^2}+\ .\ .\ .\right\}\text{where } \log\frac{2}{\gamma}=0.11593$$

which it will be noted is a real function of x, since $J_0\,(ix)$ is always real.

Rewriting equation (50) in terms of this particular form for a second solution,

$$\theta=AJ_0(i\mu)+BiH_0(i\mu) \tag{53}$$

from which

$$\frac{d\theta}{d\mu}=-AiJ_1(i\mu)+BH_1(i\mu)$$

[7] Among standard texts on Bessel functions may be cited: N. Nielsen, Handbuch der Theorie der Cylinderfunktionen-Teubner, 1904; Gray and Matthews, Treatise on Bessel Functions, MacMillan, 1895; W. E. Byerly, Fourier's Series and Spherical Harmonics, Chap. VII, Ginn, 1902; Jahnke and Emde, Funktiontafeln, Section XIII, Teubner, 1902.

[8] Hankel, Mathematische Annalen 1, p. 483, 1869; 8, p. 453, 1875.

[9] Jahnke and Emde, Funktionentafeln, p. 134 of 1909 edition.

by virtue of the properties of Bessel functions that $d\,J_0\,(x)/dx = -J_1\,(x)$ or $d\,H_0\,(x)/dx = -H_1\,(x)$. In order that equations (52) may be satisfied,

$$B = \frac{iAJ_1(i\mu_1)}{H_1(i\mu_1)} \tag{54}$$

and A must fulfill the condition $\theta = \theta_0$ when $\mu = \mu_2$.

$$\theta = \theta_0\,\frac{H_1(i\mu_1)J_0(i\mu) - J_1(i\mu_1)H_0(i\mu)}{H_1(i\mu_1)J_0(i\mu_2) - J_1(i\mu_1)H_0(iu_2)} \tag{55}$$

For the particular purpose of this section, it is not necessary to tabulate numerical values of equation (55) and map the temperature distribution in the fin. The object sought here is an expression for fin effectiveness, defined as in the preceding sections. The heat which would be dissipated by the fin shown in Figure 7 to air at temperature zero if the fin surface were all at temperature θ_0 would be (per unit length of fin per unit time)

$$2q\,\frac{w'}{\cos\alpha}\,\theta_0$$

while that actually dissipated is

$$2q\int_0^{w'}\theta\,\frac{dx}{\cos\alpha}$$

from which it follows that fin effectiveness f is

$$f = \frac{1}{w'}\int_0^{w'}\frac{\theta}{\theta_0}\,dx \tag{56}$$

From equation (48)

$$dx = \frac{\mu}{2b^2}\,d\mu$$

$$f = \frac{1}{2b^2w'}\int_{\mu_1}^{\mu_2}\frac{\theta}{\theta_0}\,\mu d\mu \tag{57}$$

There are two ways of integrating (57), which are in principal identical, and of course lead to the same result. One is to substitute for θ/θ_0 the value given by equation (55) and integrate the resulting expression by using as a reduction formula

$$\frac{d}{dx}\,[xJ_1(x)] = xJ_0(x),\ \text{and likewise for } H,$$

and the other results from noting the identity

$$\frac{d}{d\mu}\left(\mu\,\frac{d\theta}{d\mu}\right) = \mu\,\frac{d^2\theta}{d\mu^2} + \frac{d\theta}{d\mu}$$

From equation (49)

$$\mu\theta = \mu\,\frac{d^2\theta}{d\mu^2} + \frac{d\theta}{d\mu}$$

whence

$$\int\mu\theta d\mu = \int d\left(\mu\,\frac{d\theta}{d\mu}\right) = \mu\,\frac{d\theta}{d\mu}$$

Also, from (52), $\frac{d\theta}{d\mu}$ vanishes for $\mu = \mu_1$, whence

$$f = \left[\frac{1}{2b^2w'\theta_0}\,\mu_2\,\frac{d\theta}{d\mu}\right]_2 = \frac{i\mu_2}{2b^2w'}\,\frac{J_1(i\mu_1)H_1(i\mu_2) - J_1(i\mu_2)H_1(i\mu_1)}{H_1(i\mu_1)J_0(i\mu_2) - J_1(i\mu_1)H_0(i\mu_2)} \tag{58}$$

In computing numerical values with equation (58), the values of μ_1 and μ_2 may be obtained from equations (52) in terms of w', t_1, and α, the geometrical constants of the fin. Where the fin extends to a sharp-edge intersection of its two sloping surfaces, $t_1=0$ and therefore $\mu_1=0$ and $\mu_2=2b\sqrt{w'}=2\sqrt{qw'/k\sin\alpha}$. The function H_1 is infinite for zero value of the argument, whence for $\mu_1=0$ it is necessary to consider (58) in the form given below to avoid an indeterminate ∞/∞.

$$f=\frac{\mu_2}{2b^2w'}\,\frac{iJ_1(0)\,\dfrac{H_1(i\mu_2)}{H_1(0)}-iJ_1(i\mu_2)}{J_0(i\mu_2)+iJ_1(0)\,\dfrac{iH_0(i\mu_2)}{H_1(0)}}=\frac{-\mu_2}{2b^2w'}\,\frac{iJ_1(i\mu_2)}{J_0(i\mu_2)} \tag{59}$$

This special case, a fin extending to a sharp edge, could, of course, be solved very much more simply than by making it a special case of the more general problem. Recurring to equations (49) and (50), if the condition of solution to be met is $d\theta/d\mu=0$ when $\mu=0$, familiarity with Bessel functions shows at once that an abbreviated form of equation (50), namely

$$\theta=AJ_0(i\mu)$$

will meet the condition stated and at the same time have one arbitrary constant left so that the boundary condition at $x=w$ may be fulfilled. This method of procedure confirms very easily the result reached in equation (59).

For the interpretation of equation (59), substitute for μ_2 its value $2b\sqrt{w'}$, bearing in mind that for all engine cylinder fins the taper is so small that $\sin\alpha=\alpha=\tan\alpha$ within the accuracy which is required by engineering practice. The value of α is then approximately $\frac{1}{2}\,t_2/w$, and if we denote by t_m the mean thickness of the fin which tapers uniformly from t_2 at one edge to zero at the other, then $\alpha=t_m/w$. (In this case, w' and w are identical.) The reduced expression for equation (59) is

$$f=\frac{-1}{w\sqrt{\dfrac{q}{kt_m}}}\,\frac{iJ_1\left(i2w\sqrt{\dfrac{q}{kt_m}}\right)}{J_0\left(i2w\sqrt{\dfrac{q}{kt_m}}\right)} \tag{60}$$

and expresses the effectiveness of a wedge-shaped fin in terms of its physical and geometrical characteristics.

Following the general plan of this paper, the next step is to ascertain the difference between f, computed by the exact formula (60) and a value f' which would result from employing the very simple expression (5); in other words, assuming that a wedge-shaped fin of moderate taper functions nearly enough like a parallel-sided one to permit of using the formula developed for parallel sides and then making a correction for the error introduced by this procedure.

$$f'=\frac{\tanh aw'}{aw'}\quad\text{where } a=\sqrt{\frac{2q}{kt}} \tag{5}$$

Rewriting f in terms of a,

$$f=\frac{-\sqrt{2}}{aw}\;\frac{iJ_1\,(iaw\sqrt{2})}{J_0\,(iaw\sqrt{2})} \tag{61}$$

in which it is seen that f is an explicit function of the product aw just as is f', so that the two functions may be compared at any desired point. The difference is about 6 per cent for $aw=1$; about 16 per cent for $aw=2$; and 25 per cent for $aw=3$. The fin dimensions commonly employed are such that aw probably never exceeds 2 and is pretty generally less than 1. A comparison of the two functions is plotted as Figure 8.

It is reasonable to suppose that in all cases a trapezoid section fin would differ less in behavior from that of the corresponding rectangular section fin than would a triangular section fin of equivalent mean thickness and width, so that in using the simple expression f' for computing effectiveness of wedge-shaped fins, the maximum error occurring would be that corresponding to the difference between the two curves of Figure 8. It is therefore possible to use the simple formula and apply a correction with a fairly accurate degree of approximation, estimating the correction from the relative approach to a rectangle or a triangle of the fin section under consideration. A more exact but less convenient procedure is to compile a table or set of curves giving the exact correction under various circumstances. Such a set of curves forms Figure 9. The derivation is as follows:

The taper of a fin may be expressed in terms of the fin width and the ratio of its thickness at the tip (t_1, fig. 7) to its mean thickness t_m.

Let

$$\lambda = \frac{t_1}{t_m} \tag{62}$$

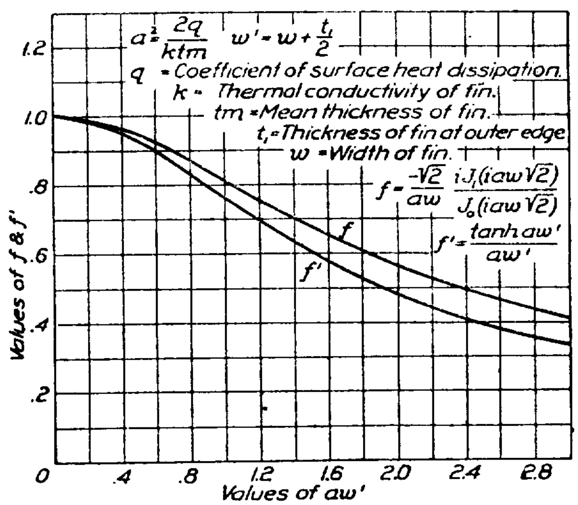


FIG. 8. Comparison of the two functions which express effectiveness of a triangular section and of a rectangular section fin. J is the functional for Bessel's function of the primary type. i is the complex symbol $\sqrt{-1}$. J_0 with an imaginary argument is a real function, and J_1 is a pure imaginary, so that iJ_1 is a real function.

Then α of Figure 7 may be expressed in terms of w, t_m, and λ and with no approximation other than $\sin \alpha = \tan \alpha$, substitution in the expressions (52) which define μ_1 and μ_2 lead to

$$\mu_1 = \frac{aw}{1-\lambda}\sqrt{\lambda}\sqrt{1-\frac{t_m}{w}(1-\lambda)} \qquad \mu_2 = \frac{aw}{1-\lambda}\sqrt{2-\lambda} \tag{63}$$

where a has the value used in all the previous developments, namely, $\sqrt{2q/kt_m}$. Since t_m/w is always small with respect to unity,

$$\mu_1 = \frac{aw}{1-\lambda}\sqrt{\lambda}\left[1-\frac{t_m}{2w}(1-\lambda)\right] \tag{64}$$

and the term in brackets can usually be omitted. There is a 1 to 1 correspondence between values of aw and approximate values of fin effectiveness, so that for any desired fin effectiveness a suitable value of aw may be read from the curve of Figure 3, and by means of the relations (63), (64), μ_1 and μ_2 may be tabulated as functions of λ for any effectiveness. Values of μ_1 and μ_2 so obtained are then substituted in equation (58) and a comparison between the resulting f and the approximate effectiveness f' will give the corrections, as a function of the taper ratio λ, which must be applied to the approximate function f'. Such curves are plotted[10] in Figure 9.

[10] The complementary procedure is to take stated values of λ and for a series of such values determine the correction as a function of approximate effectiveness. This procedure has been adopted for Figure 10, where it is only necessary to show the curves $\lambda=0$, $\lambda=0.5$ and $\lambda=0.75$ to permit of sufficiently accurate interpolation, by inspection, of any other curve of the family for the purpose of obtaining the correction to f' for any taper at any effectiveness.

4. CIRCUMFERENTIAL FINS.

For a circumferential fin of considerable width on the average size engine cylinder, it is not to be expected that relations developed for a long, rectangular fin will hold without appreciable correction. It will be shown here that the magnitude of the correction is well within the limits which justify the procedure of employing the approximate formula, corrected when necessary, in preference to using an exact solution of this problem with its attendant complications. The difference in the physical behavior of a given area of circumferential fin and of rectangular plate is perhaps most easily pictured by focusing attention on the mean circumference. When the fin width is not small with respect to the radius of curvature of this mean, there is going to be a considerable difference between the fin area, for a given length of this median, which is within the mean circumference from that area outside it, whereas in a rectangular plate, the median bisects the area. With such a difference in area distribution, it is clear that the temperature of the median is not going to be by any means equal to that which is found at the median of the rectangular plate. The use of the rectangular plate formulas is therefore more in the nature of analogy than of approximation, but it is nevertheless convenient.

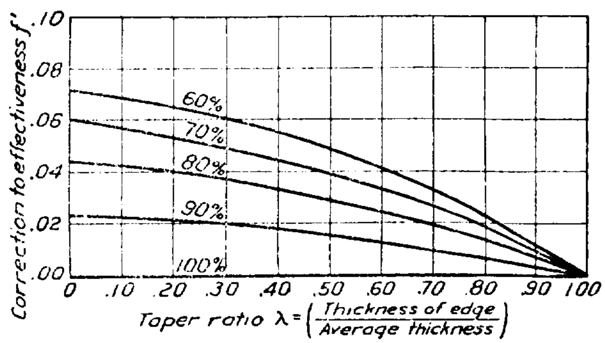


FIG. 9.—Wedge-shaped fins. Corrections to approximate effectiveness $f' = \frac{\text{Tanh } aw'}{aw'}$

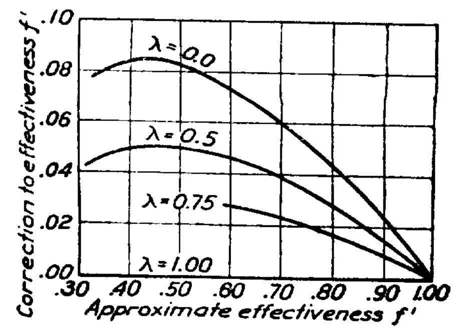


Fig. 10—Wedge-shaped fins. Corrections to f'.

A further picture which may assist in visualizing the physical processes involved comes from comparing the way in which a circumferential fin differs from a corresponding straight one to the way in which a tapering fin differs from its analogue of uniform thickness. In the latter case, we have practically identical surfaces with differences in the metallic conducting area from root to tip of fin. Since the metal near the tip is less useful, removing a certain fraction there and adding it correspondingly at the root where it is most needed results in a fin somewhat more effective than the same average thickness fin with no taper. If, now, we take a straight fin of uniform thickness and bend it around an arc, we do nothing to alter the metallic conduction process, but do change the disposition of surface. We get a less proportion of the surface in near the engine cylinder, where the temperature head is greater, and a correspondingly greater fraction out at the rim, where it is less useful. Accordingly, the curved fin is slightly less effective than its straight analogue. The possibility of certain similarities in the mathematical treatment of the two corrections, taper and curvature, thus presents itself. As a matter of fact, it turns out that the mathematical functions involved are practically identical, although leading to corrections in opposite directions, as the above picture indicates. The taper correction which has just been developed in detail and found to be always a positive correction to f' is paralleled by one for curvature, always negative.

The notation for the circumferential fins is as follows:

R_c = inner radius (namely, the outer radius of the engine cylinder).

R_f = outer radius (extreme fin radius).

t = fin thickness, assumed uniform.

$w = R_f - R_c$ (fin width).

$r, \theta,$ coordinates (polar) of any element of the fin.

The other symbols employed retain the same significance as in previous sections. The heat dissipated from the edge of the fin is to be taken into account by the method developed in section (2) of this paper, adding $t/2$ to the fin width. The exact correction, to give a surface identically half that of the edge, would be

$$\frac{t}{2}\left(1-\frac{t}{4R_t}+\ \cdot\ \cdot\ \cdot\right)$$

but the sum of the terms following 1 is usually less than 1 per cent, and $t/2$ is in itself only a small correction term, so that the omission of these terms causes no appreciable error.

The fundamental equation of heat transfer in a metal, expressed in plane polar coordinates, combined with the condition for surface dissipation from both sides of each element of fin surface, is

$$\frac{\partial^2\theta}{\partial r^2}+\frac{1}{r}\frac{d\theta}{dr}=a^2\theta \qquad a^2=\frac{2q}{kt} \tag{65}$$

(θ, temperature at any point; q, coefficient surface heat dissipation; k, thermal conductivity; t, thickness, assumed uniform). The boundary conditions are

$$\frac{\partial\theta}{\partial r}=0 \text{ when } r=R'_t=R_t+\frac{t}{2} \tag{66}$$

$$\theta=\theta_o \text{ when } r=R_c$$

Equation (65) is similar to (49), in fact identical with it if μ/a be substituted for r, whence the form of solution is given by equation (50) or equation (53), and since the boundary conditions (66) are identically those of equations (52) with appropriate values of ar instead of μ_1 and μ_2, it is unnecessary to discuss any details of solution of equation (65). The result may be taken by inspection from equation (55),

$$\theta=\theta_o\frac{H_1\,(iaR'_t)\;J_o\,(iar)-J_1\,(iaR'_t)\;H_o\,(iar)}{H_1\,(iaR'_t)\;J_o\,(iaR_c)-J_1\,(iaR'_t)\;H_o\,(iaR_c)} \tag{67}$$

Following the usual procedure for expressing the effectiveness of the fin surface, divide the heat dissipated by the fin, namely,

$$2q\int_{R_c}^{R_t'}\theta\cdot 2\pi r\cdot dr$$

by that which would be dissipated by an equal area of cylinder wall at temperature θ_o, namely,

$$2q\cdot\pi\ (R_t'^{\,2}-R_c^2)\ \theta_o \tag{68}$$

$$f=\frac{2\int_{R_c}^{R_t'} r\theta\ dr}{\theta_o\ (R_t'^2-R_c^2)}$$

and upon reducing this expression to an integrated form, employing a process identical with that used for reducing equation (57), there results as the expression for effectivemess of a circumferential fin,

$$f=\frac{2R_c}{a\ (R'^2_t-R_c^2)}\ \ \frac{i\,J_1\,(iaR_c)\;H_1\,(iaR'_t)-i\,J_1\,(iaR'_t)\;H_1\,(iaR_c)}{J_o\,(iaR_c)\;H_1\,(iaR_t')-J_1\,(iaR_t')\;H_o\,(iaR_c)} \tag{69}$$

It is convenient to have the result stated in terms of the fin width and ratio of the inner and outer fin radii.

Let

$$\rho=\frac{R_c}{R_t'} \tag{70}$$

$$w'=R_t'-R_c$$

From these definitions, it follows that

$$R_f' = \frac{w'}{1-\rho} \qquad R_o = \frac{\rho w}{1-\rho}$$

$$f = \frac{2\rho}{aw'\,(1+\rho)} \; \frac{iJ_1\left(iaw'\,\frac{\rho}{1-\rho}\right)H_1\left(\frac{iaw'}{1-\rho}\right) - iJ_1\left(\frac{iaw'}{1-\rho}\right)H_1\left(iaw'\,\frac{\rho}{1-\rho}\right)}{J_o\left(iaw'\,\frac{\rho}{1-\rho}\right)H_1\left(\frac{iaw'}{1-\rho}\right) - J_1\left(\frac{iaw'}{1-\rho}\right)H_o\left(iaw'\,\frac{\rho}{1-\rho}\right)} \tag{71}$$

Equation (71) may be used to give f as a function of aw' for any specified value of ρ, or to give f as a function of ρ for a specified value of aw', as may be desired. Thus, two processes are open to choice for the tabulations or charts to give the corrections to apply to the approximate value of fin effectiveness, defined, as previously, to be the function $f' = \tanh aw'/aw'$ to take account of the circumferential-shaped fin. In Figures 11 and 12 will be found sets of curves plotted by both processes.

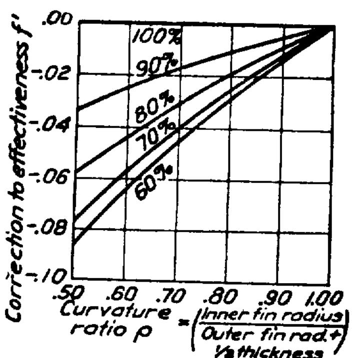


Fig. 11.—Circumferential fins. Corrections to approximate effectiveness f'.

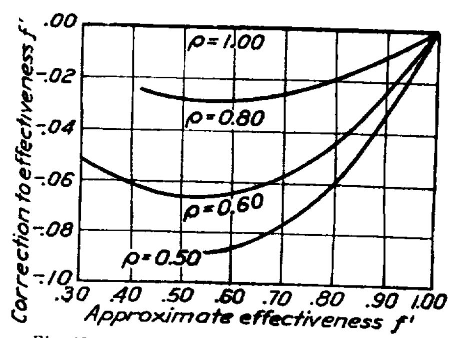


Fig. 12.—Circumferential fins. Corrections to f'.

IV. RECAPITULATION OF MATHEMATICAL DERIVATIONS—CONCLUSIONS.

Making four general assumptions of physical nature which are stated and fully discussed in the early paragraphs of Section II of this paper, it is found that the fundamental mathematics of heat conduction lead to the expression

$$f' = \frac{\tanh aw'}{aw'} \text{ where } w' = w + \frac{t}{2} \text{ and } a = \sqrt{\frac{2q}{kt}}.$$

w = width of fin.

t = thickness of fin.

q = coefficient of surface heat dissipation, units of heat per unit time per unit surface per unit difference in temperature between the fin surface and the air stream into which the heat is dissipated.

k = thermal conductivity of the fin material, units of heat per unit time per unit cross-sectional area per unit temperature gradient.

f' = fin effectiveness; the ratio of the heat dissipated by the fin to that which would be dissipated in the same time by an equal surface all at a temperature identical with that of the base of the fin; i. e., the temperature of the engine cylinder wall along the line of attachment of the fin.

The above expression for fin effectiveness would be rigorously exact under the following assumptions:

(1) The temperature across the fin thickness is constant; i. e., flow of heat is linear from base toward tip, with inappreciable "cross flow" in the direction of smallest dimension of the fin.

(2) The fin is so long with respect to other two dimensions that the heat dissipation from the exposed ends is an inappreciable fraction of the total.

(3) Conditions at the exposed edge are such that the heat loss therefrom can be accounted for by adding to the real fin width one-half of the fin thickness to get a fictitious width for use in the equations and treating the exposed edge as though perfectly blanketed.

(4) The temperature distribution prevailing at the base of the fin is that of uniform, constant temperature.

(5) The fin thickness is uniform.

(6) The fin is of rectangular contour.

By removing restrictions expressed in assumptions (1) and (3) and setting up the equations to express exactly the thermal behavior of a fin which obeys the remaining four conditions, a solution may be obtained which indicates the correction necessary to apply to the function f' quoted above to take account of the error introduced by making assumptions (1) and (3). Were this exact solution somewhat more manageable, it is obvious that proper procedure would be to employ it directly in computations rather than as a tool to construct corrections to an inexact formula. However, it is found to be extremely complicated and unsuited to repeated use. Fortunately, it proves that for any combination of geometrical and physical properties likely to characterize an engine cylinder fin, the correction will be within 1 per cent, and for the usual present-day designs it is only 0.1 or 0.2 per cent. It is therefore entirely negligible in comparison with other errors inherent in applying the mathematics to practical problems.

In view of the foregoing, it is quite justifiable for all practical applications of these mathematical developments to neglect entirely the slight discrepancy between the hypothetical conditions outlined in assumptions (1) and (3) and the real conditions which do occur. It is entirely satisfactory to employ the formula based on the assumption as a formula representing very exactly the actual fin performance. Assumption (2) is also obviously valid within satisfactory limits for any numerical work with radial fins (always long). For circumferential fins any error due to the assumption merges into that discussed in connection with assumption (6).

The limitation expressed by assumption (4) also vanishes without requiring any modification of the function f'. Provided that we can predicate the conditions outlined in the first three assumptions and assume the rectangular contour imposed by (5) and (6), it is a comparatively simple mathematical problem to derive the expression for fin effectiveness when the temperature distribution along the fin base is described by any arbitrary function of the coordinate parallel to the fin length. The result is tanh aw'/aw', or, in other words, the function already quoted is equally applicable for uniform and nonuniform base temperatures.

The assumptions numbered (5) and (6) are fulfilled by few, if any, of the fins occurring in practice, and it is therefore of prime importance to ascertain how large a deviation from the conditions described in these assumptions may occur before the use of a formula based upon them becomes absurd. The exact solution for a tapered fin which is straight in its length coordinate (i. e., a trapezoid section right prism) or for a uniformly thick circumferential fin is not a problem offering serious mathematical difficulties, nor is the result of either solution a prohibitively complicated expression for use in direct numerical application. But the functions occurring (Bessel functions of both kinds with imaginary arguments) are distinctly unfamiliar to others than specialists in mathematics, and tables of their values for numerical work are not always conveniently accessible. It has seemed very desirable, therefore, to give in this paper values of the differences between the exact solutions for these cases and the function tanh aw'/aw', so that the latter expression might always be used as the basis of a computation and corrections applied for its error.

For the trapezoid section prism fin the results are expressed in terms of the fin width (3) and a taper ratio defined as the ratio of the thickness at the tip (t_1) to that at center (t_m), a uniform taper being assumed. Designating this ratio by λ, it is $\lambda = t_1/t_m$. For a sharp-edge fin $\lambda = 0$, while the parallel surface fin has $\lambda = 1$. The fin width w is corrected to w' by adding $t_1/2$, as in the cases discussed previously. The process of computing fin effectiveness is the following: Compute the function tanh aw'/aw', using (in "a") for the value of t, the mean fin thickness t_m. Then, from this approximate value of effectiveness and the value of λ, interpolate on the family of curves forming Figure 9 or those forming Figure 10 and read off a quantity to be added to the approximate effectiveness. This will give the fin effectiveness as if computed from the exact Bessel function formula. The limitations to accuracy of the method are those imposed by the curves of Figures 9 and 10, which have been computed to the highest accuracy convenient for the tables of Bessel functions, etc., usually at hand. Generally speaking, this was within 1 part in 1,000 for each individual interpolation made, and for the result when all factors and terms are brought together it is quite certain that the corrected f' is reliable within 1 per cent, probably within a few tenths of 1 per cent.

For a circumferential fin, the process suggested is very similar to that just outlined for the correction on account of taper. Results are expressed in terms of fin width w, which is the difference between the outer and inner fin radii and a ratio ρ of the inner radius of curvature to the outer. It is obvious that for very small values of ρ, namely, the configuration approaching a circular plate with no hole ($\rho = 0$), it would be absurd to use a formula based upon a long, rectangular fin and the "corrections" to such formula which have any real validity as corrections are limited to the larger values of ρ. Between the values $\rho = 1$ and $\rho = 0.5$ the method is applicable, but toward the lower value of ρ the corrections become large and likewise less certain. After increasing w by half the fin thickness to w', compute tanh aw'/aw' and then in terms of this value of approximate effectiveness and ρ, the curvature ratio, make use of Figure 11 or 12 to ascertain the correction to be added to the approximate value to obtain the true value of fin effectiveness. It may be noted that this correction is always negative; i. e., a circumferential fin is less effective than the value computed from tanh aw'/aw'.

Summing up the foregoing paragraphs with respect to the difference between conditions which would meet assumptions (1) to (6) and the conditions which actually prevail, it is to be noted that only the differences concerned in (5) and in (6) have appreciable effect upon computations for the fins of internal-combustion engines. In case both assumptions are violated at once, namely, a tapering fin of circumferential type, two corrections may be applied to the approximate function, with a somewhat less degree of accuracy than pertains to either correction alone. The correction for taper was determined on the hypothesis of no circumferential curvature and the correction for curvature on the hypothesis of uniform thickness, whence it is clear that if both factors are concerned the corrections applied by this method are not exact The deviation is a second-order error and is usually too small to be significant in industrial applications of such a computation.

V. EXAMPLES OF COMPUTATIONS.

For the purpose of illustrating the ease of using the formulas whose derivation has been discussed above, a few examples are appended. These are selected from two well-known aviation engines—the Gnome, which has steel fins, and the Lawrance, with aluminum fins. A long, straight fin of the same width and equivalent uniform thickness is used for an initial example, followed by approximating to (*a*) the taper (*b*) the annular shape of the real engine fin. For physical interpretation of the results of the computations, the definition of effectiveness must be borne in mind. An effectiveness of 85 per cent means that each small area of fin, say, 1 cm.^{2} or 1 sq. in., is equivalent in heat dissipating power to 0.85 as much area all at a temperature identical with that of the engine cylinder wall in that vicinity.

The computations illustrate in a convincing manner the relative unimportance of high thermal conductivity for fin metal. While the difference in effectiveness of the steel and aluminum fins is quite appreciable, nevertheless it may be noted that the effectiveness of steel fins, very thin and yet reasonably wide, is high enough that for the conditions assumed in these examples it is comparable to that of aluminum fins having five times the thermal conductivity of the steel ones. Thus, no great incentive exists, on this score at least, to employ metals of very high thermal conductivity.

EXAMPLE 1.—*Steel fin, long, rectangular, uniform thickness.*

Width, 1.60 cm.

Thickness, 0.08 cm.

Thermal conductivity of steel, assume 0.10 cgs. units.

Surface dissipation coefficient—assume $q=0.003$ cgs. units, corresponding probably to a free-air speed in the vicinity of the engine of 40 to 50 meters per second, 90 to 110 miles per hour.

$$w'=w+\frac{t}{2}=1.60+0.04=1.64$$

$$a=\sqrt{\frac{2q}{kt}}=\sqrt{\frac{2\times 0.003}{0.10\times 0.08}}=\sqrt{0.75}=0.866$$

$$aw'=1.420$$

$$\tanh aw'=0.8896$$

$$f=\frac{\tanh aw'}{aw'}=0.626$$

The effectiveness of such a steel fin is thus about 63 per cent.

EXAMPLE 2.—*Steel fin, long, rectangular, tapering.*

Width 1.60 cm.

Thickness, 0.05 cm at tip and 0.11 cm at root.

Thermal conductivity of steel, assume 0.10 cgs units.

q, assume 0.003, as in Example 1.

$$w'=w+\frac{t_1}{2}=1.600+0.025=1.625 \text{ cm.}$$

$$a=\sqrt{\frac{2q}{kt_m}}=\sqrt{\frac{2\times 0.003}{0.10=0.08}}=0.868$$

$$aw'=1.408$$

$$\tanh aw'=0.8870$$

$$f=\frac{\tanh aw'}{aw'}=0.630$$

The taper factor $\lambda=t_1/t_m$ is $0.05/0.08=0.625$ and from Figure 10 the correction for a taper factor 0.63 and approximate effectiveness 0.63 is 0.03_4. This correction added to 0.630 gives 0.66_4.

The effectiveness of such a steel fin is therefore 66 per cent.

It is perhaps worth while to illustrate the computation of effectiveness of a tapering fin directly from the exact equation (58) which has furnished the basis of the corrections plotted as Figure 10.

$$f=\frac{k\alpha}{2qw'}\,\mu_2\,\frac{iJ_1\,(i\mu_1)\,H_1\,(i\mu_2)-iJ_1\,(i\mu_2)\,H_1\,(i\mu_2)}{H_1\,(i\mu_1)\,J_0\,(i\mu_2)+iJ_1\,(i\mu_1)\,iH_0\,(i\mu_2)}$$

The half angle α ($=\tan\alpha$) of the wedge is 0.0187.

From this, by equations (52)

$$\mu_1=2.90$$

$$\mu_2=4.34$$

$$J_0\,(i\mu_2)=15.17$$

$$^{11}iJ_1\,(i\mu_1)=3.613$$

$$iJ_1\,(i\mu_2)=-13.30$$

$$iH_0\,(i\mu_2)=0.00487$$

$$^{11}H_1\,(i\mu_1)=-0.0288$$

$$H_1\,(i\mu_2)=0.00540$$

$$\frac{k\alpha\mu_2}{2qw'}=0.831$$

$$f=0.831\,\frac{(-3.613)\,(-0.00540)-(-13.30)\,(-0.0288)}{(-0.0288)\,(15.17)+(-3.613)\,(0.00487)}$$

$$f=0.665$$

[11] Jahnke and Emde, Funktionentafeln, pp. 130-135, 1909 edition.

EXAMPLE 3.—*Steel fin, circumferential, thickness uniform.*

Width, 1.60 cm.
Thickness, 0.08 cm.
Inner radius, 5.65 cm.
Thermal conductivity of steel, 0.10 cgs. units.
$q=0.003$ cgs. units.

From Example 1 the first approximation to effectiveness is 0.626. The outer fin radius, 7.25 cm. plus half the thickness, is 7.29, whence the circumferential curvature factor $\rho=R_o/R'_t$ is 5.65/7.29, or 0.775.

From Figure 12 the correction for a circumference factor of 0.78 and approximate effectiveness 0.63 is -0.032. Adding this to 0.626 gives 0.594.

The effectiveness of this fin is therefore approximately 59 per cent.

To illustrate the computation of the above example directly from the exact solution for an annular fin, equation (71), instead of through the medium of a correction curve based on this equation, the following figures are summarized:

$$f=\frac{2\rho}{aw'\,(1+\rho)}\,\frac{iJ_1\left(iaw'\frac{\rho}{1-\rho}\right)H_1\left(\frac{iaw'}{1-\rho}\right)-iJ_1\left(\frac{iaw'}{1-\rho}\right)H_1\left(iaw'\frac{\rho}{1-\rho}\right)}{J_0\left(iaw'\frac{\rho}{1-\rho}\right)H_1\left(\frac{iaw'}{1-\rho}\right)+iJ_1\left(\frac{iaw'}{1-\rho}\right)iH_0\left(iaw'\frac{\rho}{1-\rho}\right)}$$

From example 1, $aw'=1.420$

$$\rho=0.775$$

$$1-\rho=0.225$$

$$aw'\frac{\rho}{1-\rho}=4.89$$

$$\frac{aw'}{1-\rho}=6.32$$

$$J_0\left(iaw'\frac{\rho}{1-\rho}\right)=24.69$$

$$iJ_1\left(iaw'\frac{\rho}{1-\rho}\right)=-22.00$$

$$iJ_1\left(\frac{iaw'}{1-\rho}\right)=-84\ \text{(estimated)}$$

$$iH_0\left(iaw'\frac{\rho}{1-\rho}\right)=0.00265$$

$$H_1\left(\frac{iaw'}{1-\rho}\right)=-0.000605$$

$$H_1\left(iaw'\frac{\rho}{1-\rho}\right)=-0.00291$$

$$\frac{2\rho}{aw'\,(1+\rho)}=0.615$$

$$f=0.615\frac{(-22.00)\ (-0.000605)-(-84)\ (-0.00291)}{(24.69)\ (-0.000605)+(-84)\ (0.00265)}=0.597$$

NOTE.—The value 84 for iJ_1 (i 6.32) can not be ascertained with any precision, but by using an identical value in both numerator and denominator, the accuracy of computation is not vitiated more than 1 or 2 parts in 500 by the probable error. The agreement of 0.594 with 0.597 is well within the limit to be expected in the use of Figure 12.

With a steel fin having the taper of Example 2 and the curvature of Example 3, with the remaining characteristics the same as those taken for all three examples, the effectiveness would be approximately 0.594 (Example 3) plus 0.034 for taper, or 0.63.

(*b*) Computation of optimum dimensions of fins [12] to meet any specified relations which are not mutually inconsistent. An example of this is minimum weight compatible with a given cooling requirement, or head resistance, or combination of the two.

In applying the equations to compute optimum dimensions, it must be borne in mind that too much must not be expected in the way of results. There are always a number of conditions to be met which are conflicting and for which a rigid mathematical specification of all would render the problem unsolvable. In fin design these include maximum cooling power, minimum weight, minimum head resistance to the air stream, adequate strength to withstand crushing under rough handling, a choice of metal and geometrical form consistent with the possibility of manufacturing with reasonable convenience and without prohibitive cost, etc. The engineer has no grounds for expecting mathematics to furnish a single inviolable solution for the optimum dimensions of the fin that will meet best such an array of specifications, but he does have the right to expect mathematics to furnish him definite relations in which he can weigh the various factors. Then, according to his judgment of relative importance of such factors, he can select the design which he considers best. It is the purpose of this paper to supply definite relations respecting the effectiveness of cooling for fins of ordinary type.

BUREAU OF STANDARDS,
Washington, D. C., December 1, 1921.

[12] Examples of optimum dimension calculation seem best deferred to a separate paper so as to permit of more detailed discussion than can be included here.

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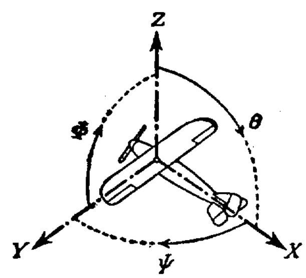

Positive directions of axes and angles (forces and moments) are shown by arrows.

Axis.		Force (parallel to axis) symbol.	Moment about axis.			Angle.		Velocities.	
Designation.	Symbol.		Designation.	Symbol.	Positive direction.	Designation.	Symbol.	Linear (component along axis).	Angular.
Longitudinal	X	X	rolling	L	$Y \longrightarrow Z$	roll	Φ	u	p
Lateral	Y	Y	pitching	M	$Z \longrightarrow X$	pitch	Θ	v	q
Normal	Z	Z	yawing	N	$X \longrightarrow Y$	yaw	Ψ	w	r

Absolute coefficients of moment

$$C_l = \frac{L}{q\,b\,S} \quad C_m = \frac{M}{q\,c\,S} \quad C_n = \frac{N}{q\,f\,S}$$

Angle of set of control surface (relative to neutral position), δ. (Indicate surface by proper subscript.)

4. PROPELLER SYMBOLS.

Diameter, D
Pitch (a) Aerodynamic pitch, p_a
(b) Effective pitch, p_e
(c) Mean geometric pitch, p_g
(d) Virtual pitch, p_v
(e) Standard pitch, p_s
Pitch ratio, p/D
Inflow velocity, V'
Slipstream velocity, V_s

Thrust, T
Torque, Q
Power, P
(If "coefficients" are introduced all units used must be consistent.)
Efficiency $\eta = T\,V/P$
Revolutions per sec., n; per min., N
Effective helix angle $\Phi = \tan^{-1}\left(\frac{V}{2\pi r n}\right)$

5. NUMERICAL RELATIONS.

1 HP = 76.04 kg. m/sec. = 550 lb. ft/sec.
1 kg. m/sec. = 0.01315 HP
1 mi/hr. = 0.44704 m/sec.
1 m/sec. = 2.23693 mi/hr.

1 lb. = 0.45359 kg.
1 kg. = 2.20462 lb.
1 mi. = 1609.35 m. = 5280 ft.
1 m. = 3.28083 ft.